*Linear and Multilinear Algebra*
Vol. 00, No. 00, Month 200x, 1–15

**RESEARCH ARTICLE**

# An extension of the polytope of doubly stochastic matrices

Richard A. Brualdi[a] and Geir Dahl[b]*

[a] Department of Mathematics, University of Wisconsin, Madison, WI 53706.
[b] Center of Mathematics for Applications, Departments of Mathematics and Informatics, University of Oslo, P.O. Box 1053 Blindern, 0316 OSLO, Norway.

(*Received 00 Month 200x; in final form 00 Month 200x*)

We consider a class of matrices whose row and column sum vectors are majorized by given vectors $b$ and $c$, and whose entries lie in the interval $[0, 1]$. This class generalizes the class of doubly stochastic matrices. We investigate the corresponding polytope $\Omega(b|c)$ of such matrices. Main results include a generalization of the Birkhoff - von Neumann theorem and a characterization of the faces, including edges, of $\Omega(b|c)$.

**Key words:** Doubly stochastic matrices, majorization, polytope, faces.
**AMS subject classifications:** 15B36, 15B51, 15A39.

## 1. Introduction

Let $\Omega_n$ denote the set of all doubly stochastic matrices of order $n$, i.e., nonnegative matrices where each row and column sum is 1. A classical theorem due to Birkhoff and von Neumann ([1], [9]) says that the extreme points of $\Omega_n$ are the permutation matrices. The purpose of this paper is to investigate a more general class of polytopes $\Omega(b|c)$ which contains $\Omega_n$ as a special case.

The underlying notion for defining $\Omega(b|c)$ is majorization. The $i$'th largest component in a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is denoted by $x_{[i]}$. If $x, y \in \mathbb{R}^n$, we say that $x$ is *majorized* by $y$, and write $x \preceq y$, whenever $\sum_{j=1}^k x_{[j]} \le \sum_{j=1}^k y_{[j]}$ for $k = 1, 2, \ldots, n$, with equality for $k = n$. Majorization plays an important role in e.g. combinatorics, statistics and matrix theory. The book [8] is a comprehensive study of majorization theory and its applications.

The object we study, $\Omega(b|c)$, is the set of all matrices $A = [a_{ij}]$ with $0 \le a_{ij} \le 1$ for each $i, j$ and whose row sum vector and column sum vector satisfy a majorization constraint. The role of majorization in connection with classes of integral matrices or $(0, 1)$-matrices is discussed in detail in [4]. A central result is the Gale-Ryser theorem which characterizes the existence of a $(0, 1)$-matrix with given row and column sums in terms of a certain majorization for these given vectors. In [5] one studies doubly stochastic matrices whose rows and columns satisfy a majorization constraint, while [6] treats the class of integral matrices with given column sums and whose rows satisfy majorization constraints.

The paper is organized as follows. Section 2 introduces the main notion, line-sum majorization and the class $\Omega(b|c)$. We relate this object to $\Omega_n$ and prove a generalization of the Birkhoff - von Neumann theorem. In Section 3 the goal is to study the facial structure of the polytope $\Omega(b|c)$. A useful connection to so-called

* Corresponding author. Email: geird@math.uio.no

ISSN: 0308-1087 print/ISSN 1563-5139 online

DOI: 10.1080/03081080xxxxxxxxx
http://www.informaworld.com

majorization polyhedra is established. Then we characterize the faces of $\Omega(b|c)$ and determine their dimensions. In particular, we determine all edges of $\Omega(b|c)$.

**Notation:** The $j$th component of a vector $x \in \mathbb{R}^n$ is denoted by $x_j$. We let $O$, $J$ and $I$ denote the all zeros matrix, the all ones matrix and the identity matrix, respectively (the dimension will be clear from the context). The $i$th unit vector in $\mathbb{R}^n$ is denoted by $e_i$. Vectors are normally treated as column vectors and are identified with the corresponding $n$-tuples. A real vector $x = (x_1, x_2, \ldots, x_n)$ is called *monotone* when $x_1 \geq x_2 \geq \cdots \geq x_n$. For an $m \times n$ matrix $A$ define its *row sum vector*

$$R(A) = (r_1(A), r_2(A), \ldots, r_m(A))$$

and *column sum vector*

$$S(A) = (s_1(A), s_2(A), \ldots, s_n(A))$$

where $r_i(A) = \sum_{j=1}^n a_{ij}$ $(i \leq m)$ and $s_j(A) = \sum_{i=1}^n a_{ij}$ $(j \leq n)$. Finally, let $\mathcal{A}(R, S)$ denote the set of $(0,1)$-matrices with row sum vector $R$ and column sum vector $S$.

## 2. Line-sum majorization

Let $b = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$ and $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ be nonnegative monotone integral vectors with

$$\tau = \sum_{i=1}^{m} b_i = \sum_{j=1}^{n} c_j.$$

Let $\Omega(b|c)$ be the set of all $m \times n$ real matrices $A = [a_{ij}]$ satisfying

$$\begin{aligned} &0 \leq a_{ij} \leq 1 \ (1 \leq i \leq m, 1 \leq j \leq n) \\ &R(A) \preceq b \\ &S(A) \preceq c. \end{aligned} \tag{1}$$

If $A \in \Omega(b|c)$, we say that $A$ is *line-sum majorized by* $(b, c)$. The $m \times n$ matrix $\overline{A} = [\overline{a}_{ij}]$ where $\overline{a}_{ij} = \frac{\tau}{mn}$ shows that, provided $0 \leq \tau \leq mn$, $\Omega(b|c)$ is always nonempty and indeed contains a positive matrix when $\tau > 0$. In that which follows, we always assume that $0 \leq \tau \leq mn$.

A special case is $b = c = e$ where $e$ is the all ones vector of length $n$. The only vector $x \in \mathbb{R}^n$ satisfying $x \preceq e$ is $x = e$. Therefore $R(A) = S(A) = e$ for each $A \in \Omega(e|e)$, and it follows that

$$\Omega(e|e) = \Omega_n$$

where $\Omega_n$ is the *Birkhoff polytope* consisting of all $n \times n$ doubly stochastic matrices. Similarly, the set of all $m \times n$ row-stochastic matrices (i.e., nonnegative matrices with each row sum being 1) is obtained as $\Omega(b|c)$ by letting $b = e$ and $c = (m, 0, 0, \ldots, 0)$. The set of column-stochastic matrices may be constructed in a similar way. We return to other examples later.

A related, but different, class of matrices was studied in [5]: the $n \times n$ matrices whose rows and columns were majorized by a fixed vector $d$. In contrast, the matrices considered here have no such constraints, but they satisfy majorization constraints on the row sum and column sum vectors. The *majorization permutahedron* $M(v) = \{x \in \mathbb{R}^N : x \preceq v\}$ was investigated in [7] and an extension of the Gale-Ryser theorem was shown. The class of integral matrices with given column sum vector and whose rows satisfy respective majorization constraints was studied in [6].

Let $x = (x_1, x_2, \ldots, x_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be such that $v$ is monotone and $\sum_{j=1}^n x_j = \sum_{j=1}^n v_j$. Then $x$ satisfies the majorization $x \preceq v$ if and only if

$$\sum_{j \in T} x_j \leq \sum_{j=1}^{|T|} v_j \quad \text{for each } T \subseteq \{1, 2, \ldots, n\}. \tag{2}$$

This follows from the fact that the maximum of the left hand side of (2), taken over such sets $T$ with $|T| = k$, equals $\sum_{j=1}^k x_{[j]}$. Thus, the set of solutions of the inequalities in (2) is precisely the majorization permutahedron $M(v) = \{x \in \mathbb{R}^n : x \preceq v\}$.

**Proposition 2.1:** *(i)* $\Omega(b|c)$ *is a bounded polyhedron in* $\mathbb{R}^{m \times n}$ *consisting of the matrices* $A = [a_{ij}]$ *satisfying*

$$\begin{array}{ll} \sum_{i \in K} \sum_{j=1}^n a_{ij} \leq \sum_{i=1}^{|K|} b_i & (K \subseteq \{1, 2, \ldots, m\}) \\ \sum_{j \in L} \sum_{i=1}^n a_{ij} \leq \sum_{j=1}^{|L|} c_j & (L \subseteq \{1, 2, \ldots, n\}) \\ \qquad 0 \leq a_{ij} \leq 1 & (1 \leq i \leq m,\ 1 \leq j \leq n). \end{array} \tag{3}$$

$\Omega(b|c)$ *is therefore a polytope and, in addition, it is invariant under row and column permutations.*

*(ii) If* $b' \preceq b$ *and* $c' \preceq c$*, then* $\Omega(b'|c') \subseteq \Omega(b|c)$.

**Proof:** (i) The first statement follows from the definition of $\Omega(b|c)$ and the inequalities (2) applied to the majorizations $R(A) \preceq b$ and $S(A) \preceq c$. Thus, $\Omega(b|c)$ is the solution set of a (finite) system of linear inequalities, so it is a polyhedron. Since $\Omega(b|c)$ clearly is bounded, polyhedral theory shows that $\Omega(b|c)$ is a polytope (the convex hull of a finite set of points). The second statement follows as row and column permutations applied to a matrix lead to permuted row and column sums, but this does not change the majorizations (as majorization is permutation invariant). Property (ii) follows from the transitivity of the majorization order. $\square$

Since $\Omega(b|c)$ is a polytope, it is natural to investigate its extreme points. We first need some results on majorization.

For monotone vectors $x = (x_1, x_2, \ldots, x_n)$ and $v = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ such that $x \preceq v$, define the *coincidence set*

$$K^{x \preceq v} = \{k \in \{1, \ldots, n\} : \sum_{j=1}^k x_j = \sum_{j=1}^k v_j\}$$

and let $K^{x \preceq v} = \{k_1, k_2, \ldots, k_p\}$ where $1 \leq k_1 < k_2 < \cdots < k_p = n$. The notion of coincidence set was used in [3] to study the doubly stochastic matrices associated with a given majorization. For $T \subseteq \{1, 2, \ldots, n\}$ we define $v_T = \sum_{j=1}^{|T|} v_j$ (which only depends on the cardinality of $T$). If $x$ is monotone, there are integers $1 \leq j_1 <$

$j_2 < \cdots < j_q = n$ for some $q \geq 1$ such that

$$x_1 = \cdots = x_{j_1} > x_{j_1+1} = \cdots = x_{j_2} > \cdots > x_{j_{q-1}} = \cdots = x_{j_q}.$$

We call $T \subseteq \{1, 2, \ldots, n\}$ an *$x$-leading subset* if, for some $s$, $T$ consists of $1, 2, \ldots, j_s$ plus possibly a subset of $\{j_s + 1, \ldots, j_{s+1}\}$. The next lemma will be useful later. Recall that $e_i$ denotes the $i$th unit vector (in $\mathbb{R}^n$).

**Lemma 2.2:** *Let $x, v \in \mathbb{R}^n$ be monotone vectors with $x \preceq v$ and let $K^{x\preceq v} = \{k_1, k_2, \ldots, k_p\}$ be as above.*
*(i) Let $T \subseteq \{1, 2, \ldots, n\}$ and define $k = |T|$, $\nu = \max\{j : j \in T\}$. Then*

$$\sum_{j\in T} x_j = v_T$$

*if and only if $T$ is an $x$-leading subset, $x_k = x_{k+1} = \cdots = x_\nu$ and $k, k+1, \ldots, \nu \in K^{x\preceq v}$.*
*(ii) Assume $k_{t-1} < i \leq j \leq k_t$ for some $t$. Then, for suitably small $\epsilon > 0$,*

$$x + \epsilon(e_i - e_j) \preceq v \quad \text{and} \quad x - \epsilon(e_i - e_j) \preceq v.$$

**Proof:** We first prove the following property.
*Claim: Assume $k \in K^{x\preceq v}$ and $x_{k+1} = x_k$. Then $k + 1 \in K^{x\preceq v}$.*
Proof of Claim: We have $\sum_{j=1}^k x_j = \sum_{j=1}^k v_j$, $\sum_{j=1}^{k+1} x_j \leq \sum_{j=1}^{k+1} v_j$ and $\sum_{j=1}^{k-1} x_j \leq \sum_{j=1}^{k-1} v_j$. This gives $x_{k+1} \leq v_{k+1}$ and $x_k \geq v_k$. Thus, as $v$ is monotone,

$$x_k \geq v_k \geq v_{k+1} \geq x_{k+1}$$

and because $x_k = x_{k+1}$, we obtain $x_k = v_k = v_{k+1} = x_{k+1}$. This gives $\sum_{j=1}^{k+1} x_j = \sum_{j=1}^{k+1} v_j$. So $k + 1 \in K^{x\preceq v}$ and the Claim follows.
(i) Now, let $T \subseteq \{1, 2, \ldots, n\}$ and assume that $\sum_{j\in T} x_j = v_T$. Define $k = |T|$ and $\nu = \max\{j : j \in T\}$. As $x \preceq v$, and both $x$ and $v$ are monotone

$$(*) \quad v_T = \sum_{j\in T} x_j \leq \sum_{j=1}^{k} x_j \leq v_T$$

and therefore both inequalities in $(*)$ hold with equality. In particular, $k = |T| \in K^{x\preceq v}$. Moreover, $x_\nu = x_k$ (otherwise the first inequality in $(*)$ would be strict). But then $x_k = x_{k+1} = \cdots = x_\nu$. By repeated application of the Claim we conclude that $k, k+1, \ldots, \nu \in K^{x\preceq v}$. It also follows from $(*)$ that $T$ must contain each $j$ for which $x_j > x_k$. We conclude that $T$ is an $x$-leading subset. Conversely, if $T$ is an $x$-leading subset and $k := |T| \in K^{x\preceq v}$, then $\sum_{j\in T} x_j = \sum_{j=1}^k x_j = \sum_{j=1}^k v_j = v_T$. This proves (i).
(ii) There is nothing to show if $i = j$, so assume that $k_{t-1} < i < j \leq k_t$ for some $t$. Note that $k_{t-1} + 1, k_t - 1 \notin K^{x\prec v}$. Therefore, by the Claim, $x_{k_{t-1}} > x_{k_{t-1}+1}$ and $x_{k_t} > v_{k_t}$. So $x_{k_t} > v_{k_t} \geq v_{k_t+1} \geq x_{k_t+1}$.
We claim that for suitably small $\epsilon > 0$ the vector $y = x + \epsilon(e_i - e_j)$ is majorized by $v$. To show this we only need to worry about the active inequalities among $\sum_{j\in T} x_j \leq v_T$ (as the strict such inequalities will still hold for $y$ with $\epsilon$ small enough). By property (i) of this Lemma, if $\sum_{j\in T} x_j = v_T$ holds, then $T$ is an $x$-leading subset and $|T| \in K^{x\preceq v}$. Therefore, as $x_{k_{t-1}} > x_{k_{t-1}+1}$ and $x_{k_t} > x_{k_t+1}$, it

follows that $T$ contains one of the two sets $\{1, 2, \ldots, k_{t-1}\}$ or $\{1, 2, \ldots, k_t\}$. Thus $T$ contains either both or none of the $i$ and $j$ chosen above, and then $\sum_{j \in T} y_j = \sum_{j \in T} x_j \leq v_T$ and it follows that $y \preceq v$. Similarly, it follows that $x - \epsilon(e_i - e_j) \preceq v$, and the proof is complete. $\square$

The next theorem determines the extreme points of $\Omega(b|c)$ and shows that this polytope is integral, i.e., it has only integral extreme points.

**Theorem 2.3:** *The extreme points of $\Omega(b|c)$ consists of all $(0,1)$-matrices $A \in \mathbb{R}^{m \times n}$ satisfying $R(A) \preceq b$ and $S(A) \preceq c$. Thus, the extreme points are all the matrices in the classes $\mathcal{A}(R, S)$ with $R \preceq b$ and $S \preceq c$.*

**Proof:** Let $A \in \Omega(b|c)$ be an extreme point of $\Omega(b|c)$. We may assume that both $R(A)$ and $S(A)$ are monotone: this is obtained by suitable row and column permutations, and, by symmetry (Proposition 2.1), these operations do not destroy the extreme point property.

Let $r_i = r_i(A)$ and $s_j = s_j(A)$ be $i$th row sum and $j$th column sum in $A$ ($1 \leq i \leq m$, $1 \leq j \leq n$). Thus,

$$r_1 \geq r_2 \geq \cdots \geq r_m \text{ and } s_1 \geq s_2 \geq \cdots \geq s_n.$$

Define the coincidence set

$$K = K^{R(A) \preceq b}$$

and let $K = \{k_1, k_2, \ldots, k_p\}$ where $1 \leq k_1 < k_2 < \cdots < k_p = m$. So, the sum of the $k$ first row sums in $A$ equal the upper bound $\sum_{i=1}^{k} b_i$ precisely when $k \in K$. Similarly, define the coincidence set

$$L = K^{S(A) \preceq c}$$

and let $L = \{l_1, l_2, \ldots, l_q\}$ where $1 \leq l_1 < l_2 < \cdots < l_q = n$. The sum of the $l$ first column sums in $A$ equals the upper bound $\sum_{j=1}^{l} b_j$ precisely when $l \in L$. We may partition $A$ according to $K$ and $L$ as follows

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ & & \vdots & \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{bmatrix}$$

where the $t$'th block row corresponds to rows $k_{t-1} + 1, \ldots, k_t$ in $A$ and the $v$'th block column corresponds to columns $l_{v-1} + 1, \ldots, l_v$ in $A$, where $t \leq p$, $v \leq q$ and we define $k_0 = l_0 = 0$. The sum of all entries in the $t$'th block row equals

$$\sum_{i=k_{t-1}+1}^{k_t} b_i$$

and sum of all entries in the $v$'th block column equals

$$\sum_{j=l_{v-1}+1}^{l_q} b_j.$$

Note that all these sums are integers. Our goal is to prove that $a_{ij} \in \{0,1\}$ for each $1 \le i \le m$, $1 \le j \le n$. Define $F = \{(i,j) : 0 < a_{ij} < 1\}$, the positions of the "fractional entries" in $A$. Assume that $F$ is nonempty; we shall derive a contradiction from this. Let $(i_1, j_1) \in F$, say $k_{t-1} < i_1 \le k_t$ so $(i_1, j_1)$ is in the $t$'th block row.

Since

$$\sum_{i=k_{t-1}+1}^{k_t} r_i = \sum_{i=k_{t-1}+1}^{k_t} b_i$$

which is an integer (as $b$ is integral), there must exist $(i_2, j_2) \in F$ such that $(i_1, j_1) \neq (i_2, j_2)$ and $(i_2, j_2)$ lies in the same block row of $A$ and, say, $l_{v-1} < j_2 \le l_v$. Then

$$\sum_{j=l_{v-1}+1}^{l_v} c_j = \sum_{j=l_{v-1}+1}^{l_v} b_j$$

which is an integer, so there is fractional entry $a_{i_3 j_3}$ in the same block column of $A$. We may continue like this and, eventually, we get a cycle (after possible reordering)

$$(i_1, j_1), (i_2, j_2), \ldots, (i_h, j_h), (i_1, j_1)$$

where all these positions lie in $F$. For $\epsilon \in \mathbb{R}$ let $A^\epsilon = [a^\epsilon_{ij}]$ be the matrix obtained from $A$ by letting $a^\epsilon_{i_t j_t} = a_{i_t j_t} + \epsilon$ when $t$ is even, and and $a^\epsilon_{i_t j_t} = a_{i_t j_t} - \epsilon$ when $t$ is odd.

*Claim: For suitably small $\epsilon > 0$ both $A^\epsilon$ and $A^{-\epsilon}$ lie in $\Omega(b|c)$.*

Proof of Claim: Clearly, for small $\epsilon > 0$ we have $0 \le a^\epsilon_{ij} \le 1$ and $0 \le a^{-\epsilon}_{ij} \le 1$ for each $i, j$. The majorizations $R(A^\epsilon) \preceq b$ and $S(A^\epsilon) \preceq c$ as well as $R(A^{-\epsilon}) \preceq b$ and $S(A^{-\epsilon}) \preceq c$ now follow from Lemma 2.2. This proves the Claim.

Finally, $A = (1/2)A^\epsilon + (1/2)A^{-\epsilon}$ holds, so by the Claim this contradicts that $A$ is an extreme point of $\Omega(b|c)$. This proves that $F = \emptyset$ so each entry in $A$ is 0 or 1. The theorem now follows (the second statement in the theorem follows directly from the first statement). $\square$

It is easy to construct $(0,1)$-matrices in $\Omega(b|c)$, that is, extreme points. Due to Theorem 2.3, the extreme points are all the matrices in the classes $\mathcal{A}(R,S)$ with $R \preceq b$ and $S \preceq c$. So, for instance, take $R$ to be an $m$-vector with as nearly equal as possible entries summing to $\tau$ and similarly for $S$. It is easy to verify that $S \preceq R^*$, so by the Gale-Ryser theorem (see e.g. [4]), $\mathcal{A}(R,S) \neq \emptyset$. Moreover, $R \preceq b$ and $S \preceq c$. Thus any matrix in $\mathcal{A}(R,S)$ lies in $\Omega(b|c)$. For instance, let $b = (4,4,2,0)$ and $c = (5,2,1,1,1)$. Define $R = (3,3,2,2)$ and $S = (2,2,2,2,2)$, so $R \preceq b$ and $S \preceq c$. Then the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

lies in $\mathcal{A}(R,S)$ and therefore also in $\Omega(b|c)$.

The previous theorem generalizes the classical Birkhoff - von Neumann theorem for doubly stochastic matrices (see [4] for a discussion of this result and majorization).

**Corollary 2.4:** ([1], [9])[The Birkhoff - von Neumann theorem] *The extreme points of the set $\Omega_n$ of doubly stochastic matrices are the permutation matrices.*

**Proof:** Let $b = c = e$ in Theorem 2.3. Then, as explained above, $\Omega(e|e) = \Omega_n$, and since all extreme points are integral, the extreme points must be the permutation matrices. $\square$

Let $A$ be a $(0,1)$-matrix. A *Ryser interchange* in $A$ is to replace a $2 \times 2$ submatrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

or vice versa.

The next theorem says that the extreme points of $\Omega(b|c)$ are connected using certain interchanges.

**Theorem 2.5:** *For each pair of $(0,1)$-matrices $A$ and $B$ in $\Omega(b|c)$ (i.e., extreme points) it is possible to transform $A$ into $B$ by a sequence of operations of one of the following two types*

(*i*) *a Ryser interchange*
(*ii*) *interchanging a $0$ and a $1$ lying in the same row or column.*

**Proof:** Let $A, B \in \Omega(b|c)$. First, if $A$ and $B$ both lie in the same class $\mathcal{A}(R,S)$ for some $R$ and $S$, it follows from the Ryser interchange theorem (see [4]) that we can transform $A$ into $B$ by a sequence of operations of type (i).

So, assume that $A$ and $B$ do not lie in the same Ryser class. Let (as above) $R$ be the monotone $m$-vector with as nearly equal as possible entries summing to $\tau$ and similarly for $S$. Then $\mathcal{A}(R,S)$ is nonempty, so let $C \in \mathcal{A}(R,S)$. Say that $A$ does not lie in $\mathcal{A}(R,S)$ and let the row and column sum vectors of $A$ be $R'$ and $S'$, respectively. Then there are transfers (see [8]) for the majorizations $R \preceq R'$ and $S \preceq S'$ and corresponding matrix operations of type (i) or (ii) above which transform $A$ into a matrix $A'$ in $\mathcal{A}(R,S)$. Then we can transform $A'$ further into $C$ using Ryser interchanges. In a similar way we can go from $B$ to $C$. The theorem now follows. $\square$

## 3. Faces

In this section we study the faces of $\Omega(b|c)$. Recall that a *face* $\mathcal{F}$ of a convex set $\mathcal{C} \subseteq \mathbb{R}^{m \times n}$ satisfies (i) $\mathcal{F} \subseteq \mathcal{C}$, (ii) $\mathcal{F}$ is convex, and (iii) every line segment $[A,B] = \{(1-\lambda)A + \lambda B : 0 \le \lambda \le 1\}$ in $\mathcal{C}$ with an interior point in $\mathcal{F}$ also satisfies $A, B \in \mathcal{F}$. A useful result says that, if $\mathcal{C}$ is a polyhedron in $\mathbb{R}^{m \times n}$ (so it is defined by a finite linear system of inequalities), then $\mathcal{F}$ is a nonempty face of $\mathcal{C}$ if and only if it is the solution set of the given system of linear inequalities, but where some of the inequalities are replaced by equalities. For general theory of faces of convex sets and polyhedra, see [10], [11], [13].

We first determine the dimension of $\Omega(b|c)$. Recall that $\tau = \sum_i b_i = \sum_j c_j$.

**Lemma 3.1:** *If $\tau = 0$ or $\tau = mn$, then $\Omega(b|c)$ consists only of a single matrix, namely $O_{m,n}$ or $J_{m,n}$, respectively. If $0 < \tau < mn$, then the dimension $d$ of $\Omega(b|c)$*

*is given as follows.*

(i) *If* $b_1 > b_m$ *and* $c_1 > c_n$*, then* $d = mn - 1$*.*
(ii) *If* $b_1 > b_m$ *and* $c_1 = c_n$*, then* $d = (m-1)n$*.*
(iii) *If* $b_1 = b_m$ *and* $c_1 > c_n$*, then* $d = m(n-1)$*.*
(iv) *If* $b_1 = b_m$ *and* $c_1 = c_n$*, then* $d = (m-1)(n-1)$*.*

**Proof:** The case where $\tau$ is 0 or $mn$ is clear, so assume $0 < \tau < mn$. Let $\bar{A} = [\bar{a}_{ij}]$ be the $m \times n$ matrix where each entry is $\tau/mn$. Then $\bar{A} \in \Omega(b|c)$ and $0 < \bar{a}_{ij} < 1$ for each $i, j$. Moreover, $r_i(\bar{A}) = \tau/m$ and $s_j(\bar{A}) = \tau/n$ for each $i$ and $j$.

(i) Let $T$ be a nonempty strict subset of $\{1, 2, \ldots, m\}$ and let $k = |T|$. Then

$$\sum_{i \in T} r_i(\bar{A}) = k\tau/m = k(1/m)\sum_{i=1}^{m} b_i < k(1/k)\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} b_i.$$

The strict inequality here follows from the fact that $b$ is monotone and $b_1 > b_m$. Similarly one proves that $\sum_{j \in T} s_j(\bar{A}) < \sum_{j=1}^{|T|} c_j$ for nonempty strict subsets $T$ of $\{1, 2, \ldots, n\}$. We conclude that the only inequality in (3) that holds with equality for $\bar{A}$ is $\sum_{i,j} a_{ij} = \tau$. This shows that the dimension of $\Omega(b|c)$ is at least $mn - 1$. It cannot be more since the $(m, n)$-entry is always determined by the other entries.

(ii) If $c_1 = c_n$, then the column sums of all matrices in $\Omega(b|c)$ equal $c_1$. Thus the first $n - 1$ entries in each row determine uniquely that last entry. Now proceed similar to the above. (iii) Similar to case (ii).

(iv) If $b_1 = b_m$ and $c_1 = c_n$, then all the row sums of all matrices in $\Omega(b|c)$ equal $b_1$ and all the column sums equal $c_1$. All the remaining majorization inequalities in (3) are then redundant, and due to the matrix $\bar{A}$ with entries strictly between 0 and 1, we conclude that the dimension of $\Omega(b|c)$ is $(m-1)(n-1)$. $\square$

The next theorem describes the faces of the majorization permutahedron $M(v) = \{x \in \mathbb{R}^n : x \preceq v\}$. This result will be used below to study the faces of $\Omega(b|c)$. Recall the notation $v_T = \sum_{j=1}^{|T|} v_j$ for a monotone vector $v \in \mathbb{R}^n$. Define $N_n = \{1, 2, \ldots, n\}$. Associated with $T \subseteq N_n$ is its *incidence vector* which is the $(0, 1)$-vector of length $n$ whose support equals $T$.

**Theorem 3.2:** *Let* $\emptyset \neq T_1 \subset T_2 \subset \cdots \subset T_k = N_n$ *for some positive* $k$*. Then*

$$\mathcal{F} = \{x \in M(v) : \sum_{j \in T_t} x_j = v_{T_t} \;\; (1 \le t \le k)\} \tag{4}$$

*is a face of* $M(v)$*. Conversely, every nonempty face* $\mathcal{F}$ *of* $M(v)$ *may be written in the form* (4) *for a suitable chain of subsets* $\emptyset \neq T_1 \subset T_2 \subset \cdots \subset T_k = N_n$*.*

**Proof:** Recall the inequality description of $M(v)$ given in (2). In general, each face of a polyhedron is obtained by replacing some of its defining inequalities by the corresponding equations. This implies the first part of the theorem, by setting inequalities $\sum_{j \in T} x_j \le v_T$ to equality for $T \in \{T_1, T_2, \ldots, T_k\}$.

In order to prove the converse statement we first show that the set function $T \to v_T$ is submodular, i.e.,

$$v_{T \cup T'} + v_{T \cap T'} \le v_T + v_{T'} \quad (T, T' \subseteq N_n).$$

In fact, let $T, T' \subseteq N_n$, and define $p_1 = |T \cap T'|$, $p_2 = \min\{|T|, |T'|\}$, $p_3 =$

$\max\{|T|, |T'|\}$ and $p_4 = |T \cup T'|$. So $p_1 \le p_2 \le p_3 \le p_4$ and $p_4 - p_3 = p_2 - p_1$. Then

$$(v_T + v_{T'}) - (v_{T\cup T'} + v_{T\cap T'}) = \textstyle\sum_{j=p_1+1}^{p_2} v_j - \sum_{j=p_3+1}^{p_4} v_j.$$

Since $v$ is monotone, $v_{p_1+k} \ge v_{p_3+k}$ for $k = 1, 2, \ldots, p_2 - p_1 (= p_4 - p_3)$. Thus, the right-hand side of the inequality above is nonnegative, and the submodularity follows.

Let now $\mathcal{F}$ be a nonempty face of $M(v)$. Thus, there is a family $\mathcal{T}$ of nonempty subsets of $N_n$ such that

$$\mathcal{F} = \{x \in M(v) : \sum_{j\in T} x_j = v_T \ \ (T \in \mathcal{T})\}. \tag{5}$$

We may assume that $\mathcal{T}$ is maximal with this property (so $\mathcal{T}$ contains each $T$ such that $\sum_{j\in T} x_j = v_T$ holds for all $x \in M(v)$).

*Claim 1: If $T$, $T' \in \mathcal{T}$, then $T \cap T'$, $T \cup T' \in \mathcal{T}$. So, $\mathcal{T}$ is closed under union and intersection.*

Proof of Claim 1: Assume $T, T' \in \mathcal{T}$ and let $x \in \mathcal{F}$. Then, by submodularity

$$\begin{aligned}
\textstyle\sum_{j\in T} x_j + \sum_{j\in T'} x_j &= \textstyle\sum_{j\in T\cup T'} x_j + \sum_{j\in T\cap T'} x_j \\
&\le v_{T\cup T'} + v_{T\cap T'} \\
&\le v_T + v_{T'} \\
&= \textstyle\sum_{j\in T} x_j + \sum_{j\in T'} x_j
\end{aligned}$$

so there must be equality throughout. In particular, $\sum_{j\in T\cup T'} x_j = v_{T\cup T'}$ and $\sum_{j\in T\cap T'} x_j = v_{S\cap T}$. Since $x$ was an arbitrary element in $\mathcal{F}$, we conclude that $T \cup T'$, $T \cap T' \in \mathcal{T}$; this proves the Claim.

Note that $N_n \in \mathcal{T}$ (as each $x \in M(v)$ satisfies $\sum_{j=1}^n x_j = \sum_{j=1}^n v_j$). Consider the following procedure. First, choose $T_1 \in \mathcal{T}$ such that no $T' \in \mathcal{T}$ is strictly contained in $T_1$. Then, inductively, assuming $T_1, T_2, \ldots, T_{t-1}$ have been chosen, let $T_t \in \mathcal{T}$ be such that no set $T' \in \mathcal{T}$ satisfies

$$\bigcup_{p=1}^{t-1} T_p \subset T' \subset T_t.$$

where the inclusions are strict. Continue this process until we obtain $T_k = N_n$. This gives a chain of sets $\emptyset \ne T_1 \subset T_2 \subset \cdots \subset T_k = N_n$ where $T_t \in \mathcal{T}$ $(t = 1, 2, \ldots, k)$.

We now prove that (4) holds.

*Claim 2: Let $T \in \mathcal{T}$. Then the incidence vector of $T$ (in $N_n$) is a linear combination of the incidence vectors of $T_i$ $(i \le k)$.*

Proof of Claim 2: Assume that the incidence vector of $T$ is *not* a linear combination of the incidence vectors of $T_i$ $(i \le k)$. This implies that $T$ cannot be a linear combination of the incidence vectors of the sets

$$T_1,\ T_2 \setminus T_1,\ T_3 \setminus T_2, \ldots,\ T_k \setminus T_{k-1}$$

and, in particular, $T$ cannot be a union of some of these subsets. Therefore there is a smallest possible $t$ such that $T \cap T_t$ is a *strict* subset of $T_t$. This contradicts our construction of the chain $T_1, T_2, \ldots, T_k$ since we then should have selected $T \cap T_t$, and not $T_t$, at stage $t$ of the construction procedure above. This proves Claim 2, by contradiction.

Assume that $x \in M(v)$ satisfies

$$\sum_{j \in T_t} x_j = v_{T_t} \quad (1 \leq t \leq k). \tag{6}$$

Let $T \in \mathcal{T}$. From Claim 2 it follows that the equation $\sum_{j \in T} x_j = v_T$ is implied by the the $k$ equations (6). This shows the inclusion

$$\mathcal{F} = \{x \in M(v) : \sum_{j \in T} x_j = v_T \ \ (T \in \mathcal{T})\} \supseteq \{x \in M(v) : \sum_{j \in T_t} x_j = v_{T_t} \ \ (1 \leq t \leq k)\}.$$

Since the opposite inclusion is trivial, we have shown (4), and the proof is complete. □

Theorem 3.2 says that the faces of $M(v)$ may be viewed as the direct sum of smaller majorization permutahedra obtained by decomposing $v$. Moreover, the dimension of the face $\mathcal{F}$ (given in the theorem) is at most $n-k$, and it is equal to $n-k$ if and only if $k$ is the maximal length of a chain of $T_t$'s satisfying (4). We remark that the majorization permutahedron $M(v)$ may be seen as the base polytope of the *polymatroid* associated with the submodular function $T \to v_T$, and a similar result to Theorem 3.2 holds for polymatroids (see [12]).

Note that the equations in (4) may be written equivalently as

$$\sum_{j \in T_t \setminus T_{t-1}} x_j = v_{T_t} - v_{T_{t-1}} \ \ (1 \leq t \leq k) \tag{7}$$

where $T_0 := \emptyset$ (and $v_\emptyset = 0$). Since the sets $T_1,\ T_2 \setminus T_1,\ T_3 \setminus T_2, \ldots,\ T_k \setminus T_{k-1}$ are a partition of $N_n$, the equations in (7) are constraints on the sum of components of $x$ for each set in this partition.

We now turn to $\Omega(b|c)$. Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$. Consider an ordered partition $\mathcal{K} = (K_1, K_2, \ldots, K_p)$ of $I$. Let $\kappa_t = \sum_{j=1}^t |K_j|$ for $t = 1, 2, \ldots, p$, $\kappa_0 = 0$, and define

$$b_t^{\mathcal{K}} = \sum_{j=\kappa_{t-1}+1}^{\kappa_t} b_j \ \ (t = 1, 2, \ldots, p).$$

Similarly, for an ordered partition $\mathcal{L} = (L_1, L_2, \ldots, L_q)$ of $J$ let $\gamma_t = \sum_{j=1}^t |L_j|$ for $t = 1, 2, \ldots, q$, $\gamma_0 = 0$ and $c_t^{\mathcal{L}} = \sum_{j=\gamma_{t-1}+1}^{\gamma_t} c_j$ $(t = 1, 2, \ldots, q)$. For instance, if $m = 6$, $p = 3$, $K_1 = \{2, 4, 5\}$, $K_2 = \{1, 3\}$, $K_3 = \{6\}$, then $\kappa_1 = 3$, $\kappa_2 = 5$, $\kappa_3 = 6$ and

$$b_1^{\mathcal{K}} = b_1 + b_2 + b_3, \ \ b_2^{\mathcal{K}} = b_4 + b_5, \ \ b_3^{\mathcal{K}} = b_6.$$

The following result characterizes the faces of $\Omega(b|c)$.

**Theorem 3.3:**
*Let $\mathcal{F}$ be a face of $\Omega(b|c)$. Then there are two disjoint subsets $Z_0$ and $Z_1$ of $I \times J$, an ordered partition $\mathcal{K} = (K_1, K_2, \ldots, K_p)$ of $I$ and an ordered partition*

*$\mathcal{L} = (L_1, L_2, \ldots, L_q)$ of $J$ such that $\mathcal{F}$ consists of the matrices $A \in \Omega(b|c)$ satisfying*

$$\begin{array}{rl} \sum_{i\in K_t}\sum_{j=1}^{n} a_{ij} = b_t^{\mathcal{K}} & (1 \le t \le p) \\ \sum_{j\in L_t}\sum_{i=1}^{m} a_{ij} = c_t^{\mathcal{L}} & (1 \le t \le q) \\ a_{ij} = 0 & ((i,j) \in Z_0) \\ a_{ij} = 1 & ((i,j) \in Z_1). \end{array} \tag{8}$$

*Conversely, if $\mathcal{F}$ consists of those matrices $A \in \Omega(b|c)$ satisfying (8) for some $\mathcal{K}$, $\mathcal{L}$, $Z_0$ and $Z_1$, then $\mathcal{F}$ is a face of $\Omega(b|c)$.*

**Proof:** Let $\mathcal{F}$ be a face of $\Omega(b|c)$. Since $\Omega(b|c)$ is a polyhedron, $\mathcal{F}$ is obtained from the inequality description (3) of $\Omega(b|c)$, by replacing a subset of the inequalities by the corresponding equations. Such a subsystem contains equations $a_{ij} = 0$ for $(i,j)$ in some subset $Z_0$ of $I \times J$, and $a_{ij} = 1$ for $(i,j)$ in a subset $Z_1$ of $I \times J$. Clearly, these sets $Z_0$ and $Z_1$ must be disjoint. The remaining equations in the subsystem come from the majorization constraints for row and column sums, i.e., the first two sets of inequalities in (3).

Consider first these equations for row sums, i.e.,

$$\sum_{i\in T}\sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{|T|} b_i \qquad (T \in \mathcal{T}) \tag{9}$$

where $\mathcal{T}$ is some class of subsets of $I$. Since each $A \in \Omega(b|c)$ satisfies $R(A) \preceq b$, we may consider the corresponding majorization permutahedron $M(b) = \{x \in \mathbb{R}^m : x \preceq b\}$. The equations (9), viewed as equations involving the row sums $r_i(A)$, define a face of this polytope. By Theorem 3.2, and its proof, the equations (9) are equivalent to a subset of these equations corresponding to a certain chain of subsets

$$\emptyset \neq T_1 \subset T_2 \subset \cdots \subset T_p = \{1, 2, \ldots, m\}$$

Moreover (as remarked after Theorem 3.2), by defining $K_t = T_t \setminus T_{t-1}$ for $t = 1, 2, \ldots, p$ (and $T_0 = \emptyset$) this subsystem is equivalent to

$$\sum_{i\in K_t}\sum_{j=1}^{n} a_{ij} = b_t^{\mathcal{K}} \qquad (1 \le t \le p)$$

where $\mathcal{K}$ is the ordered partition $\mathcal{K} = (K_1, K_2, \ldots, K_p)$.

Next, consider the equations from (3) corresponding to column sums. Using similar arguments as above these equations are equivalent to

$$\sum_{j\in L_t}\sum_{i=1}^{m} a_{ij} = c_t^{\mathcal{L}} \qquad (1 \le t \le q).$$

This shows that a general face $\mathcal{F}$ of $\Omega(b|c)$ has the form described in the theorem. Conversely, if $\mathcal{F}$ is a subset of $\Omega(b|c)$ satisfying the equations in (8) for some $\mathcal{K}$, $\mathcal{L}$, $Z_0$ and $Z_1$, then $\mathcal{F}$ must be a face of $\Omega(b|c)$ (as some defining inequalities have been set to equality).

□

Note that the constraints described in (8) resemble those known for transportation polytopes (see [4]); we return to this below. The difference is that here we have a constraint on the sum of certain subsets of rows (or columns).

We now investigate edges of $\Omega(b|c)$, i.e., one-dimensional faces of that polytope, and for this we introduce a certain directed graph.

Let $P = [p_{ij}]$ and $Q = [q_{ij}]$ be two distinct vertices of $\Omega(b|c)$. Let $\mathcal{F}_{P,Q}$ be the smallest face of $\Omega(b|c)$ containing $P$ and $Q$. Then $\mathcal{F}_{P,Q}$ has the form described in Theorem 3.3 for suitable ordered partitions $\mathcal{K} = (K_1, K_2, \ldots, K_p)$, $\mathcal{L} = (L_1, L_2, \ldots, L_q)$ and sets $Z_0$ and $Z_1$. We may assume that both $p$ and $q$ are largest possible. Define $B_{tt'} = K_t \times L_{t'}$ ($t \leq p$, $t' \leq q$). Then these "blocks" $B_{tt'}$ are a partition of $I \times J$. Define a directed graph $D_{P,Q}$ as follows. It contains vertices $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_q$; define $U = \{u_1, u_2, \ldots, u_p\}$ and $V = \{v_1, v_2, \ldots, v_q\}$. For each $(i,j)$ with $p_{ij} = 1$ and $q_{ij} = 0$ the pair $(i,j)$ is contained in a unique block $B_{tt'}$ and we introduce an associated arc $e_{ij}$ in $D_{P,Q}$ such that $e_{ij}$ goes from $u_t$ to $v_{t'}$. Similarly, for each $(i,j)$ with $p_{ij} = 0$ and $q_{ij} = 1$ we introduce an arc $e_{ij}$, but now it is directed from $v_{t'}$ to $u_t$. Clearly, each of these mentioned pairs $(i,j)$ is outside $Z_0 \cup Z_1$. Note that $D_{P,Q}$ may have parallel arcs and that its arc set is nonempty.

**Theorem 3.4:** *Let $P$ and $Q$ be vertices of $\Omega(b|c)$. Then $P$ and $Q$ are adjacent on $\Omega(b|c)$ if and only if the arc set of $D_{P,Q}$ is a directed cycle.*

**Proof:** Assume first that $[P,Q]$ is an edge of $\Omega(b|c)$ and consider the associated directed graph $D_{P,Q}$.
*Claim* 1*: The arc set of $D_{P,Q}$ may be partitioned into arc-disjoint* (*even*) *cycles.*
Proof of Claim 1: Both $P$ and $Q$ satisfy the equations (from (8))

$$\sum_{i \in K_t} \sum_{j=1}^{n} a_{ij} = b_t^{\mathcal{K}} \quad (1 \leq t \leq p).$$

In this equation, for fixed $t$, insert first $P$ and then $Q$, and subtract. This gives that

$$|\{(i,j) \in N_1 : i \in K_t\}| = |\{(i,j) \in N_2 : i \in K_t\}|.$$

Thus, the indegree and the outdegree in $D_{P,Q}$ coincide at each vertex $v_t$. Similarly, using equations for column sums in (8), we conclude that the indegree and the outdegree coincide at each vertex $u_t$, so this holds for all vertices in $D_{P,Q}$. But this implies that the arc set of $D_{P,Q}$ may be partitioned into arc-disjoint (even) cycles (this is a directed version of Veblen's theorem, see [2]). Each such cycle is even as $D_{P,Q}$ is (directed) bipartite. This proves Claim 1.

We now strengthen Claim 1.
*Claim* 2*: The arc set of $D_{P,Q}$ is precisely a directed cycle.*
Proof of Claim 2: Consider Claim 1 and assume that $D_{P,Q}$ contains two such arc-disjoint cycles $C_1$ and $C_2$. Let $C_1'$ be obtained from $C_1$ be reversing the direction of each arc in $C_1$. As a result we obtain a matrix $P'$ which agrees with $P$ in all positions except for those positions corresponding to arcs in $C_1$, and in those positions 1's and 0's have been interchanged compared to $P$. Note that we only modified entries outside $Z_0 \cup Z_1$ and that $P'$ has the same number of 1's in each block row and block column defined by the partitions $\mathcal{K}$ and $\mathcal{L}$. This implies, by Lemma 2.2, that $R(P') \preceq b$ and $S(P') \preceq c$, so $P' \in \Omega(b|c)$. Moreover, $P'$ satisfies all the equations (8) so $P' \in \mathcal{F}$. But $P'$ is different from $P$ and $Q$, so the edge $\mathcal{F}$ contains *three* vertices; a contradiction. This proves Claim 2.

Claim 2 shows the first part of the theorem. Conversely, if the arc set of $D_{P,Q}$ is a directed cycle, then it is easy to see that $\mathcal{F}_{P,Q}$ cannot contain any other $(0,1)$-matrices than $P$ and $Q$, which implies that $[P,Q]$ is an edge.

$\square$

**Example:** Let $b = (2,0)$ and $c = (1,1)$. Note that $(1,1) \preceq b$. Then the only $(0,1)$-matrices in $\Omega(b|c)$ are

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } S = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Consider faces corresponding to different ordered partitions $(\mathcal{K},\mathcal{L})$ as in the theorem above. Note that we must have $\mathcal{L} = (\{1\},\{2\})$ as $c = (1,1)$.

- Let $\mathcal{K} = (\{1\},\{2\})$. Then each matrix in $\mathcal{F}$ satisfies $r_1(A) = 2$, so the only matrix in is $\mathcal{F}$ is $R$. Similarly, if $\mathcal{K} = (\{2\},\{1\})$, then each matrix in $\mathcal{F}$ satisfies $r_2(A) = 2$, so the only matrix in is $\mathcal{F}$ is $S$.
- Let $\mathcal{K} = (\{1,2\})$ so $p = 1$. Then $\mathcal{F}$ equals $\Omega(b|c)$ if $Z_0 = Z_1 = \emptyset$. This implies that the smallest face $\mathcal{F}_{P,Q}$ containing $P$ and $Q$ is $\Omega(b|c)$, so $[P,Q]$ is not an edge. Similarly, we conclude that $[R,S]$ is not an edge. These facts can also be seen directly from Theorem 3.4 because the digraph $D_{P,Q}$ contains two arc-disjoint cycles: $D_{P,Q}$ is given by $U = \{u_1\}$, $V = \{v_1, v_2\}$ and arcs $(u_1,v_1)$, $(v_1,u_1)$, $(u_1,v_2)$, $(v_2,u_1)$. In contrast, $[P,R]$ is an edge; then $D_{P,R}$ contains the arcs $(u_1,v_2)$ and $(v_2,u_1)$, so precisely one cycle. Similarly, we see that $[R,Q]$, $[Q,S]$ and $[S,P]$ are edges. So $\Omega(b|c)$ is a rectangle with corners at $P,Q,R,S$ and diagonals $[P,Q]$ and $[R,S]$.

$\square$

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be monotone nonnegative integral vectors, and let $\widetilde{\mathcal{A}}(R,S)$ be the *truncated transportation polytope* consisting of all nonnegative matrices $A = [a_{ij}]$ with row and column sum vectors $R$ and $S$, respectively, where $0 \le a_{ij} \le 1$ for all $1 \le i \le m, 1 \le j \le n$. Assume that $\widetilde{\mathcal{A}}(R,S)$ is nonempty. A face $\mathcal{F}$ of $\widetilde{\mathcal{A}}(R,S)$ consists of all matrices in this polytope where a certain specified set $S_0$ of entries have been set equal to 0 and a disjoint set $S_1$ of entries have been set equal to 1, in such a way that there is at least one matrix in $\widetilde{\mathcal{A}}(R,S)$. Thus a face is specified by a $(0,1,*)$-matrix $F$ where $*$ means that no explicit restriction is imposed on an entry. We may assume that for each entry $(i,j)$ of $F$ equal to $*$, there is a matrix in $\mathcal{F}$ whose $(i,j)$-entry is strictly between 0 and 1; otherwise, we may replace the $*$ in $F$ with 0 or 1 without changing the face specified by $F$. (Note: if there is a matrix in the face whose $(i,j)$-entry is 0 and one whose $(i,j)$-entry is 1, then by taking convex combinations, we see that there is a matrix in the face whose $(i,j)$-entry is any specified number between 0 and 1.) Such a matrix $F$ is called $*$*-minimal* for $\widetilde{\mathcal{A}}(R,S)$. We set $\sigma_*(F)$ equal to the number of entries of $F$ equal to $*$. If some row or column contains a $*$, then, since $R$ and $S$ are integral, it must contain at least one other $*$.

Let $F = [f_{ij}]$ be a $(0,1,*)$-matrix of size $m \times n$. Consider the complete bipartite graph $K_{m,n}$ with vertex set $\{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_n\}$ and edges $[u_i, v_j]$ $(i \le m,\, j \le n)$. Let $G_*(F)$ be the subgraph of $K_{m,n}$ induced by the edges $[u_i, v_j]$ for which $f_{ij} = *$ $(i \le m,\, j \le n)$. The matrix $F$ is $*$*-indecomposable* provided that $G_*(F)$ is connected. In what follows we assume that the matrix $F$ specifying $\mathcal{F}$ is $*$-indecomposable, as the general case follows from the $*$-indecomposable case using additivity of the dimension as usual.

**Theorem 3.5:** *If the face $\mathcal{F}$ of $\widetilde{\mathcal{A}}(R,S)$ is specified by a $*$-minimal, $*$-indecomposable matrix $F$, then the dimension of $\mathcal{F}$ equals*

$$\sigma_*(F) - m - n + 1.$$

**Proof:** Let $F = [f_{ij}]$. For each $(i,j)$ such that $f_{ij} = *$, there is a matrix in $\mathcal{F}$ whose $(i,j)$-entry is strictly between 0 and 1. By taking convex combinations, there is a matrix $X = [x_{ij}] \in \mathcal{F}$ such that $0 < x_{ij} < 1$ for *all* $(i,j)$ with $f_{ij} = *$. Since $G_*(F)$ is connected, it has a spanning tree $T_*$. Let $W$ be the set of entries of $F$ (necessarily equal to $*$) which correspond to the edges of $T_*$. It follows inductively, using the fact that a tree has a pendent vertex, that a matrix in $\mathcal{F}$ is uniquely determined once the entries corresponding to the entries of $F$ outside $W$ but equal to $*$ have been specified. Since $T_*$ has $m+n-1$ edges, it follows that the dimension of $\mathcal{F}$ is at most equal to $\sigma_*(F) - (m+n-1)$.

Now consider the matrix $X$ defined above. For each $(i,j)$ with $0 < x_{ij} < 1$ such that the $(i,j)$-entry of $F$ does not correspond to an edge of $T_*$, there is a small interval $I_{ij}$ containing $x_{ij}$ such that for each of the matrices $Z = [z_{ij}]$ such that $z_{ij} = f_{ij}$ if $f_{ij} = 0$ or 1, $z_{ij} = 0$ if $(i,j)$ corresponds to an edge of $T$, and $z_{ij} \in I_{ij}$ otherwise, the row and column sum vectors are dominated by $R$ and $S$, respectively. We may then specify inductively the entries of $Z$ corresponding to the edges of $T_*$ and obtain a matrix in $\mathcal{F}$. It follows that $\mathcal{F}$ contains a rectangular parallelepiped of dimension $\sigma_*(F) - (m+n-1)$, and thus the dimension of $\mathcal{F}$ is as given in the statement of the theorem. $\square$

Now let $\mathcal{F}$ be a face of $\Omega(b|c)$ and use the notation of Theorem 3.3 and its proof. The face $\mathcal{F}$ can be regarded as a *block* truncated transportation polytope specified by a block matrix $F$ as in Theorem 3.3. A block can contain only specified entries (0 or 1) or it can contain at least one entry equal to $*$. We refer to those blocks that contain at least one entry equal to $*$ as a $*$-*block*. The block matrix $F$ is $*$-*block indecomposable* provided the bipartite graph $G_{b*}(F) \subseteq K_{p,q}$ whose edges correspond to the blocks containing at least one entry equal to $*$ is connected. As above, we assume that $G_{b*}(F)$ is connected. We also assume that the matrix $F$ is $*$-minimal, meaning as above, that replacing any $*$ with either a 0 or a 1 results in a proper subface of $\mathcal{F}$. The quantity $\sigma_*(F)$ is the number of entries equal to $*$ (and not the number of blocks containing an entry equal to $*$). Note that because of the integrality of the vectors $b$ and $c$, if some row or column of blocks contains a $*$, then it must contain at least one other $*$. With these assumptions we have the following theorem.

**Theorem 3.6:** *Let $\mathcal{F}$ be a face of $\Omega(b|c)$ specified by $F$ as above. Then the dimension of $\mathcal{F}$ equals*

$$\sigma_*(F) - p - q + 1.$$

**Proof:** This may be shown very similar to the proof of Theorem 3.5 with the graph $G_{b*}(F)$ replacing the graph $G_*(F)$. One takes a spanning tree $T_{b*}$ of $G_{b*}(F)$ and proceeds with arguments as above. The only difference is that in the blocks corresponding to the edges of $T_{b*}$ one can vary in an interval all but one of the entries corresponding to the $*$s, and use the remaining $*$ in the block to achieve the desired block row and column sums. $\square$

We now consider the 1-dimensional faces of $\Omega(b|c)$, that is, the edges of the vertex-edge graph of this polytope. In order that a face $\mathcal{F}$ be an edge, $\sigma_*(F) - p - q + 1$

must equal 1, that is,

$$\sigma_*(F) = p + q.$$

Then the graph $G_{b*}(F) \subseteq K_{p,q}$, being connected, has at least $p+q-1$ edges. Thus either (I) there are $p+q$ blocks each containing one (and only one) $*$, or (II) there are $p+q-1$ blocks containing one $*$, with exactly one of these blocks containing two $*$s.

First consider case I. Then $G_{b*}(F)$ has exactly $p+q$ edges, and since it has $p+q$ vertices, $G_{b*}(F)$ is a unicyclic connected graph. If $G_{b*}$ had a vertex of degree one, then we contradict the maximality property of $F$ (the entry corresponding to that $*$, being the only unspecified entry either in its row or column block would have to be 0 or 1). Thus $G_{b*}(F)$ is a cycle through all the vertices and hence $p = q$. It follows that $\mathcal{F}$ has two vertices obtained by alternating 0s and 1s in the positions corresponding to the edges of $G_{b*}(F)$.

Now consider the case II. Then $G_{b*}(F)$ is a tree. It follows inductively as before that we contradict the maximality property of $F$ unless this tree has only two vertices, that is, $G_{b*}(F) \subseteq K_{1,1}$. But then $p = q = 1$ and there is only one block, and it contains exactly two entries equal to $*$. It follows that $\mathcal{F}$ contains two vertices obtained by setting these entries equal to 0 and 1, and 1 and 0, respectively.

The above characterization of faces of dimension 1 and their corresponding vertices is equivalent to that given in Theorem 3.4.

## References


[1] G. Birkhoff, Three observations on linear algebra, *Univ. Nac. Tacumán Rev. Ser. A* 5 (1946), pp. 147–151.

[2] J.A. Bondy, U.S.R. Murty, *Graph Theory*, Springer, 2008.

[3] R.A. Brualdi, The doubly stochastic matrices of a vector majorization, *Linear Algebra Appl.* 61 (1984), pp. 141–154.

[4] R.A. Brualdi, *Combinatorial Matrix Classes*, Cambridge University Press, Cambridge, 2006.

[5] R.A. Brualdi, G. Dahl, Majorization-constrained doubly stochastic matrices, *Linear Algebra Appl.* 361 (2003), pp. 75–97.

[6] R.A. Brualdi, G. Dahl, Majorization classes of integral matrices, To appear in *Linear Algebra Appl.*

[7] G. Dahl, Majorization permutahedra and $(0,1)$-matrices, *Linear Algebra Appl.* 432 (2010), pp. 3265–3271.

[8] A.W. Marshall, I. Olkin, B.C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, Springer, 2011.

[9] J. von Neumann, A certain zero-sum two-person game equivalent to the optimal assignment problem, *Contributions to the Theory of Games, vol. II* (ed. by H. W. Kuhn and A. W. Tucker), *Ann. of Math. Studies*, no. 28, Princeton, 1953, pp. 5–12.

[10] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[11] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, 1998.

[12] A. Schrijver, *Combinatorial Optimization - Polyhedra and Efficiency*, Springer, 2003.

[13] R. Webster, *Convexity*, Oxford University Press, Oxford, 1994.